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1999 J. Phys. A: Math. Gen. 32 L207

(http://iopscience.iop.org/0305-4470/32/18/102)

J. Phys. A: Math. Gen. **32** (1999) L207–L213. Printed in the UK PII: S0305-4470(99)01170-1

LETTER TO THE EDITOR

The long delayed solution of the Bukhvostov–Lipatov model

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Received 21 January 1999

Abstract. The solution of the Bukhvostov–Lipatov model is completed by computing the physical excitations and their factorized S matrix. The origin of the paradoxes which led in recent years to the suspicion that the model may not be integrable is also explained.

In a famous paper of 1981, Bukhvostov and Lipatov (BL) [1] presented pieces of the solution to a very interesting quantum field theory model, whose *bare* Lagrangian reads

$$\mathcal{L} = \bar{\psi}_1(\mathrm{i}\partial\!\!\!/ - m)\psi_1 + \bar{\psi}_2(\mathrm{i}\partial\!\!\!/ - m)\psi_2 + g_b\bar{\psi}_1\gamma_\mu\psi_1\bar{\psi}_2\gamma^\mu\psi_2. \tag{1}$$

Except for the massive Thirring model, most known integrable models with several fermion species do not contain an explicit mass term: the Gross Neveu models, the $SU(2)$ and $U(1)$ Thirring model etc all exhibit spontaneous mass generation. The model (1) is quite different in nature and it had not been clear until that paper what sort of 'family' of integrable theories it belonged to.

In their original paper, BL succeeded in diagonalizing the model (1) with the coordinate Bethe ansatz method. They only worked out half of the solution however, i.e. building the ground state, but stopped short of discussing the physical excitations. Although there are well established methods to do so in principle [2, 3], in practice, the Bethe equations written by BL are complex, leading to a bewildering array of excitations, where it is hard to identify fundamental particles, bound states, and 'pseudo-particles' arising from nondiagonal scattering. Certainly, the lack of a quantum group understanding of the equations in [1] did not help in that endeavour.

In the last few years, a growing suspicion has mounted that maybe the BL model was not solvable after all. When naively bosonizing (1), one obtains a double sine–Gordon model which indeed appears to be nonintegrable classically [4]. Worse, this bosonized model also appears to be nonintegrable (this aspect has again been emphasized very recently in [5]) within the framework [6] of conformal perturbation theory. The purpose in this letter is to show the simple way out of these difficulties, and to complete the solution of the BL model by computing the physical excitations and their factorized S matrix.

To start, I recall the Bethe ansatz solution of [1]. Writing the energy

$$E = \sum_j m \sinh y_j \tag{2}$$

0305-4470/99/180207+07$19.50

the equations quantizing allowed momenta are

$$
\begin{aligned}
&\exp(-\mathrm{i}mL\sinh y_j) = -\prod_{r=1}^{l}\frac{\sinh\left(y_j - z_r + \mathrm{i}\frac{g}{2}\right)}{\sinh\left(y_j - z_r - \mathrm{i}\frac{g}{2}\right)} \\
&\prod_{j=1}^{n}\frac{\sinh(z_s - y_j + \mathrm{i}\frac{g}{2})}{\sinh(z_s - y_j - \mathrm{i}\frac{g}{2})} = -\prod_{r=1}^{l}\frac{\sinh(z_s - z_r + \mathrm{i}g)}{\sinh(z_s - z_r - \mathrm{i}g)}.
\end{aligned} \tag{3}
$$

In the latter equations, I have slightly changed the notation as compared with [1]. I have introduced the coupling g, whose exact relation to the bare coupling g_b in (1) depends on the regularization used in solving the coordinate Bethe ansatz equations (it is, quite confusingly, denoted by the same symbol in [1] however!). Compared with equations (81), (82) in [1], I have also switched the sign of the coupling constant, and will moreover restrict myself to the case $g > 0$ here (that is, $g < 0$ in notations of [1]). I have set their $v_r = z_r - \frac{\mathrm{i}g}{2}$. Finally, I have chosen antiperiodic boundary conditions for the fermions. If I call N_i the number of fermions of type i, then $N_2 = l$ and $N_1 = n - l$ in (3).

While BL used a sharp cut-off regularization, I would like to proceed slightly differently, and introduce the cut-off Λ at this early stage, as was done for the Thirring model in [7]. One has sometimes to be careful with the choice of cut-off: for instance, it leads to drastic differences in the physical properties of the Thirring model in the repulsive regime [7, 8]. I have however checked in this particular case that whatever procedure gives identical results; the reason why I want to use the smooth cut-off of [7] is to facilitate later comparisons with lattice models. I will thus replace the left-hand side of the first equation in (3) as

$$
\exp(-\mathrm{i}mL\sinh y) \to \left[\frac{\sinh\frac{1}{2}\left(y_j - \Lambda + \mathrm{i}\frac{g}{2}\right)}{\sinh\frac{1}{2}\left(y_j - \Lambda - \mathrm{i}\frac{g}{2}\right)}\frac{\sinh\frac{1}{2}\left(y_j + \Lambda + \mathrm{i}\frac{g}{2}\right)}{\sinh\frac{1}{2}\left(y_j - \Lambda - \mathrm{i}\frac{g}{2}\right)}\right]^{L/2a}. \tag{4}
$$

When Λ is large, a (having the units of length) is small and $y \ll \Lambda$, this reproduces the previous term, together with the correspondence

$$
m = 2\frac{\mathrm{e}^{-\Lambda}}{a}\sin\frac{g}{2} \tag{5}
$$

while divergences are smoothly cut-off at large values of y. I will similarly use for the energy the derivative of the momentum read off from (4), that is

$$
m\cosh y_j \to \frac{1}{2}\sin\frac{g}{2}\left[\frac{1}{\cosh(y_j + \Lambda) - \cos\frac{g}{2}} + \frac{1}{\cosh(y_j - \Lambda) - \cos\frac{g}{2}}\right]. \tag{6}
$$

I can now proceed and study the physics encoded in these equations. As pointed out in [1], it is easy to check that the ground state is obtained by filling up a sea of y_j antistrings, together with a sea of real (one-string) z_r. A first possible strategy now is to study the possible physical excitations obtained by making holes, adding other types of strings etc, and to try to extract their S matrix. This turns out to be a rather confusing task however, for a reason that we will understand easily later: whatever g, the scattering is not diagonal, and there is a complex spectrum both of bound states and pseudo particles.

To make progress, I will instead use the approach pioneered by Pearce and Klümper [9], and, independently, by Destri and de Vega [10], which has since been widely used to tackle theories with complicated scattering [11]. After the usual manipulations, the KPDDV equations read

$$
\begin{aligned}
f(y) &= \mathrm{i}L2M\cos\frac{\pi g}{2}\sinh y + 2\mathrm{i}\int \mathrm{d}y'\,\Phi_{11}(y-y')\mathrm{Im}\,\ln(1+\mathrm{e}^{-f(y'-\mathrm{i}0)}) \\
&\quad +2\mathrm{i}\int \mathrm{d}z\,\Phi_{12}(y-z)\mathrm{Im}\,\ln(1+\mathrm{e}^{-g(z-\mathrm{i}0)}) \\
g(z) &= \mathrm{i}LM\sinh z + 2\mathrm{i}\int \mathrm{d}z'\,\Phi_{22}(z-z')\mathrm{Im}\,\ln(1+\mathrm{e}^{-g(z'-\mathrm{i}0)}) \\
&\quad +2\mathrm{i}\int \mathrm{d}y\,\Phi_{12}(z-y)\,\mathrm{Im}\,\ln(1+\mathrm{e}^{-f(y-\mathrm{i}0)}).
\end{aligned}
\tag{7}
$$

where I have set $M = \frac{m}{\cos\frac{\pi g}{2}}$. The energy then reads

$$
\begin{aligned}
E = \frac{L}{\pi}\bigg[& 2M\cos\frac{\pi g}{2}\int \mathrm{d}y\,\sinh y\mathrm{Im}\,\ln(1+\mathrm{e}^{-f(y-\mathrm{i}0)}) \\
& +M\int \mathrm{d}z\,\sinh z\,\mathrm{Im}\,\ln(1+\mathrm{e}^{-g(z-\mathrm{i}0)})\bigg].
\end{aligned}
\tag{8}
$$

In (7) and (8), the integrals are running from $-\infty$ to ∞. In these equations, the kernels are given by, setting $g = \frac{2\pi}{t}$, t a real number,

$$
\begin{aligned}
\hat{\Phi}_{11} &= \frac{\sinh\frac{(t-2)x}{2}}{\sinh\frac{(t+2)x}{2}} \\
\Phi_{22} &= \frac{\sinh^2 x}{\sinh\frac{(t-2)x}{2}\sinh\frac{(t+2)x}{2}} \\
\Phi_{12} &= \frac{\sinh\frac{tx}{2}}{\sinh\frac{(t+2)x}{2}}
\end{aligned}
\tag{9}
$$

where we have introduced the Fourier transform $\hat{f}(x) = \frac{1}{2\pi}\int f(y)\mathrm{e}^{\mathrm{i}xy/\pi}$.

Two things can be rigorously deduced from these equations. The first is that the UV limit ($m \to 0$) of the theory has a central charge $c = 2$, as expected from (1). The second is that the physical mass is simply proportional to the bare mass, not a power of it: this means that in the physical (renormalized) theory the BL equations are describing (I will return to this issue later), the operator perturbing the UV fixed point must have scaling dimensions $x = 1$, irrespective of g.

Besides, extracting the scattering theory from the KPDDV equations is still a matter of guess work. So far, these equations have had the very simple feature that they contain only the most ‘basic’ ingredients of the theory: the fundamental particles and their S matrix. In the sine–Gordon case for instance [11], the right-hand side would involve only one type of terms (only one distribution function), with kernel $\Phi = \frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}y}\ln S_{++}$, S_{++} the soliton–soliton scattering matrix, y the rapidity.

In this case, I claim that we can interpret the equations as follows. First, I observe that, writing

$$
\frac{\sinh^2 x}{\sinh\frac{(t-2)x}{2}\sinh\frac{(t+2)x}{2}} = \frac{\sinh x}{2\cosh\frac{tx}{2}\sinh\frac{(t-2)x}{2}} - \frac{\sinh x}{2\cosh\frac{tx}{2}\sinh\frac{(t+2)x}{2}}
$$

we can identify the Φ_{22} terms in the KPDDV equations as $\Phi_{22} = \frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}y}\ln[S_{++}^{\hat{\beta}_1}S_{++}^{\hat{\beta}_2}]$. Here, I have introduced the two parameters (recall $g = \frac{2\pi}{t}$)

$$\begin{aligned} \hat\beta_1^2 &\equiv 4\pi \frac{t-2}{t-1} \\ \hat\beta_2^2 &\equiv 4\pi \frac{t+2}{t+1} \end{aligned} \tag{10}$$

while by S^β I denote the S matrix for an ordinary sine–Gordon theory whose parameter is β [12] (that is, the dimension of the perturbing operator is $x = \frac{\beta^2}{4\pi}$):

$$S^\beta_{++}(y) = \exp\left[\mathrm{i}\int_{-\infty}^{\infty} \frac{\mathrm{d}\xi}{2\xi} \sin\frac{2y\xi\mu}{\pi} \frac{\sinh(\mu-1)\xi}{\sinh\xi\cosh\mu\xi}\right] \qquad \mu = \frac{8\pi-\beta^2}{\beta^2}. \tag{11}$$

This leads to my basic guess: the physical theory is made up of four particles (kinks) of mass M carrying a *pair* of quantum numbers $Q_1, Q_2 = \pm 1$ (how these are related to the original problem will be discussed soon), and which scatter with the S matrix

$$S = S^{\hat\beta_1} \otimes S^{\hat\beta_2}. \tag{12}$$

Notice that $\hat\beta_1^2 < 4\pi$ while $\hat\beta_2^2 > 4\pi$ for $t \in [2,\infty)$. Therefore, while the second S matrix in (12) is in the repulsive regime, the first one is in the attractive regime, and therefore will exhibit bound states. In the KPDDV equations, *neutral* bound states do not show up. Here however, because there are two charges, it is reasonable that the 'basic charged' bound states should also appear. In fact, one easily checks that the second mass in our equations, $2M\cos\frac{\pi g}{2}$, is precisely the mass for the first bound state in a SG theory with scattering $S^{\hat\beta_1}$. Moreover, one can also check that the kernels Φ_{11} and Φ_{12} are the exact kernels one would obtain when scattering one of our bound states with either another bound state, or a basic kink. In doing this check, one should not forget that, although $S^{\hat\beta_2}$ has no bound state, it of course *does* contribute to the overall scattering of bound states. For instance, Φ_{12} arises from the scattering matrix $S^{\hat\beta_1}_{b+}(y)S^{\hat\beta_2}_{++}(y-\frac{\mathrm{i}\pi}{t})S^{\hat\beta_2}_{++}(y+\frac{\mathrm{i}\pi}{t})$, where S_{b+} is the soliton one-breather S matrix in the usual sine–Gordon model with parameter $\hat\beta_1$.

The claim (12) therefore appears to be at least reasonable from that perspective. What I have done next is go backwards, and have checked it carefully against the Bethe equations by using the more traditional method of identifying the basic excitations and computing their scattering. Equation (12) turns out to be perfectly confirmed. I found out, in particular, that making a hole in the z distribution produces a fundamental kink, while making a hole in the y distribution produces a fundamental bound state (observe this is true provided $t < \infty$, that is $g > 0$. The free limit is singular from the point of view of the Bethe equations, which does not help in analysing them). Other bound states are obtained by complex distributions of roots (in particular 'strings over strings') which do not seem that interesting to discuss here. An additional result (which could have also been obtained by adding magnetic fields in the KPDDV equations) is the charge of the fundamental kinks in terms of the original charges: one has

$$\begin{aligned} Q_1 &= \frac{\beta_2^2}{2\pi}(N_1+N_2) \\ Q_2 &= \frac{\beta_1^2}{2\pi}(N_1-N_2). \end{aligned} \tag{13}$$

Here, I have introduced still other parameters, for a reason that will hopefully become clear soon:

$$\frac{\beta_1^2}{2\pi} \equiv \frac{\hat{\beta}_1^2}{8\pi - \hat{\beta}_1^2} = \frac{t-2}{t}$$
$$\frac{\beta_2^2}{2\pi} \equiv \frac{\hat{\beta}_2^2}{8\pi - \hat{\beta}_2^2} = \frac{t+2}{t}. \tag{14}$$

Notice that as $t \to \infty$, $Q_1 \to N_1 + N_2$, $Q_2 \to N_1 - N_2$, so the basic particles coincide with the four elementary fermions in that limit.

At this stage, the alert reader will have no doubt recognized that the scattering theory we have extracted from the BL equations is nothing but the scattering theory for the double sine–Gordon model [13] (a particular case of a general model studied by Fateev [14]), whose *renormalized* Lagrangian reads, after bosonization

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \partial^\mu \phi_1 + \tfrac{1}{2}\partial_\mu \partial^\mu \phi_2 + \Lambda \cos \beta_1 \phi_1 \cos \beta_2 \phi_2 \tag{15}$$

with

$$\beta_1^2 + \beta_2^2 = 4\pi. \tag{16}$$

The notations are of course chosen so that (16) matches (14). In particular, the conditions (16) mean that the dimension of the perturbing operator in (15) is $x = 1$.

In fact, this result is not so surprising. It is easy to check, at least if one requires the existence of a conserved quantity of spin three, that the only nontrivial manifolds where the double sine–Gordon model is quantum integrable are given by (16) and, maybe, the other manifold $\beta_1^2 + \beta_2^2 = 4\pi$.

From that perspective, however, the naively bosonized theory associated with (1) corresponds to $\frac{1}{\beta_1^2} + \frac{1}{\beta_2^2} = \frac{1}{\pi}$, and looks completely baffling! What happens is, I think, quite simple. In the coordinate Bethe ansatz, one always deals with a *bare* Lagrangian, that is then regularized by using a particular prescription: making sense of terms like $\delta(x)\mathrm{sign}(x)$ when one solves the Bethe equations, and introducing a cut-off in the rapidity integrals. There is of course no reason why the resulting large distances properties should be described by a renormalized theory whose parameters are the same as the bare ones! This well known fact is hammered home by the example of the ordinary Thirring model (with four fermion coupling g_T): Korepin for instance [2] found that $\beta^2 = 4(\pi - g_T)$, (Bergknoff and Thacker [15] have a different result) which differs from Coleman's [16] famous correspondence, $\beta^2 = \frac{4\pi}{1+\frac{g_T}{\pi}}$. None of this is in the least suprising of course, and it did not matter very much until now, because the models one was dealing with were always integrable anyway; it was just a matter of knowing which particular point in one language corresponds to which particular point in the other.

Things are very different here, since the models of interest are integrable only in a subset of the whole parameter space: Lagrangians have to be specified much more carefully, and the integrable theory might look quite different depending on which point of view one adopts. In bosonization, as well as in conformal perturbation theory, one usually deals with *renormalized* theories. There is thus no reason why, by naively bosonizing the bare Lagrangian of BL and interpreting it at face value for a renormalized Lagrangian, one should find a theory that is integrable in conformal perturbation theory. In other words, the bare Lagrangian they wrote, together with the regularization they used, defined an integrable quantum field theory, and in that sense, the BL model is integrable†. It was, however, misleading of the authors in [1] to proceed with bosonization, and the final double sine–Gordon model they wrote down, with $\frac{1}{\beta_1^2} + \frac{1}{\beta_2^2} = \frac{1}{\pi}$, cannot be expected to be integrable.

† Classical integrability, as checked in [5], also follows.

The only proper way to proceed, once faced with (1), is to identify the scattering theory by studying the bare and physical Bethe ansatz equations. Once the S matrix (12) is obtained, one can for instance observe that it has affine quantum group symmetry $\hat{s}l_{q_1}(2) \otimes \hat{s}l_{q_2}(2)$ [13]; this, together with the fact that the dimension of the perturbing operator is $x = 1$, leads unambiguously to (15) with (16). After refermionization, (15) reads as (1) with the additional appearance of terms $(\bar{\psi}_i \gamma_\mu \psi_i)^2$. These terms come with a coefficient of order g_b^2 at small g_b—as in the Thirring model, renormalization effects are seen only at higher orders, and at leading order in the bare coupling constant all models are equivalent.

To make things more concrete and somewhat more rigorous, I would like to finally point out that the problem I have been discussing can be studied quite explicitly with a lattice model regularization. As discovered in [17], the following Hamiltonian, obtained from a twisted $Osp_q(2/2)^{(2)} R$ matrix [18], is exactly solvable:

$$
\begin{aligned}
H = \sum_{j,\sigma} &(c^\dagger_{j,\sigma} c_{j+1,\sigma} + cc)(1 - n_{j,-\sigma} - n_{j+1,-\sigma} - \sigma V_1 (n_{j,-\sigma} - n_{j+1,-\sigma})) \\
&+ V_2 \sum_j (c^\dagger_{j,+} c^\dagger_{j,-} c_{j+1,-} c_{j+1,+} - c^\dagger_{j,+} c^\dagger_{j+1,-} c_{j+1,+} c_{j,-} + cc) \\
&+ V_2 \sum_j (n_{j,+} n_{j,-} + n_{j+1,+} n_{j+1,-} + n_{j,+} n_{j+1,-} + n_{j,-} n_{j+1,+} - n_j - n_{j+1} + 1)
\end{aligned}
\tag{17}
$$

where $n_{j,\sigma} = c^\dagger_{j,\sigma} c_{j,\sigma}$, $V_1 = \sin\gamma$, $V_2 = \cos\gamma$, $g = \mathrm{e}^{\mathrm{i}\gamma}$. By using the well known techniques (see e.g. [19]) to take the continuum limit of this model at small V_1, V_2 (both negative) and half filling, and keeping only the relevant or marginal terms, I have found that (17) exactly gives rise to (1), with $g_b \propto -V_2$ and $m = 0$ (the complicated fine tuning in (17) cancels out the terms that would induce a gap in the similar looking Hubbard model). Of course, there is an infinity of additional irrelevant couplings, that will give rise to renormalized coupling constants: this is very similar to what happens in the XXZ model say, but here, this renormalization changes the form of the naive interaction quite a bit. By using the Bethe ansatz equations written in [17], I have checked that the CFT associated with (17) corresponds to the $\Lambda \to 0$ limit of (15). It is in fact possible to also put a mass term in the lattice model by using an inhomogeneous distribution of spectral parameters as in [20]; the bare Hamiltonian looks then as (1), while the Bethe equations are identical with those I used before (Λ being then, as in [20], a measure of the inhomogeneity, and a the lattice spacing).

In conclusion, it is a bit disappointing to realize that we have only one integrable manifold in the double sine–Gordon model†, the appealing but mysterious one hinted at in [1] finally coinciding, after proper analysis, with the one in [13, 14]. On the other hand, I hope that this discussion will led to further progress in understanding theories with several bosons. As far as the $O(3)$ sigma model is concerned, it seems that the Bukhvostov Lipatov approach was almost right after all; according to recent work of Al Zamolodchikov [21], the proper theory describing the instantons anti-instantons interaction differs from (15), (16), by the simple replacement $\beta_2 \to \mathrm{i}\beta_2$.

This work was supported by the DOE and the NSF (under the NYI program). I thank A Leclair, V Korepin, F Lesage, B Mac Coy, M Martins and Al Zamolodchikov for discussions.

† There are growing indications that the manifold $\beta_1^2 + \beta_2^2 = 8\pi$ is not integrable; note that the conformal perturbation theory argument is rather weak in that case, due to the operator having dimension one.

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